Quiz #1

Date: September 12, 2025

Question 1. 5 points

Identify the error in the following proof, and provide a counterexample for the wrong claim.

Proposition: The number $\sqrt{15}$ is irrational.
Proof: Observe that $\sqrt{15} = \sqrt{3}\sqrt{5}$. Now, as 3 and 5 are prime, we know that $\sqrt{3}$ and $\sqrt{5}$ are irrational, as proven in class. Thus, $\sqrt{15}$ being the product is also irrational. ■

Solution: The error is in the final claim, which implicitly claims that if x and y are irrational numbers, then the product xy is also irrational. This claim is indeed false, as can be seen by the following (counter)example: $\sqrt{2}$ is irrational, but $\sqrt{2} \cdot \sqrt{2} = 2$ is rational.

Rubric:

- **3 points:** identifying the incorrect step,
- **2 points:** providing a counterexample.

Question 2. 10 points

Let q be a prime number.

1. **(3 points)** Recall the precise definition of v_q.

Recall the following fact from class: if x is an integer, then $q^n \mid x$ if and only if $v_q(x) \geqslant n$.

2. **(6 points)** Use this fact from class to prove the following: if x and y are integers, then

$$v_q(x+y) \geqslant \min(v_q(x), v_q(y)).$$

Solution:

1. For a non-zero rational number $x \in \mathbb{Q}$, we proved in class that we may write

$$x = \pm \prod_{p \in \mathbb{P}} p^{e_p},$$

where $e_p \in \mathbb{Z}$ and $e_p = 0$ for all but finitely many p, and that such an expression is unique. We then defined $v_q(x) = e_q$. If $x = 0$ we set $v_q(x) = \infty$.

2. Assume first that x and y are both non-zero. Then, by the fact from class we have $q^{v_q(x)} \mid x$ and $q^{v_q(y)} \mid y$. So, if $m = \min\big(v_q(x), v_q(y)\big)$ then q^m divides $q^{v_q(x)}$ and $q^{v_q(y)}$

and thus q^m divides both x and y. Thus, q^m divides $x+y$. So, the fact from class implies $v_q(x+y) \geqslant m = \min\big(v_q(x), v_q(y)\big)$.

Assume now that $x = 0$. Then, as $v_q(x) = \infty$ we see that $\min\big(v_q(x), v_q(y)\big) = v_q(y)$ and so our desired inequality reduces to $v_q(y) \geqslant v_q(y)$. The case when $y = 0$ follows by symmetry.

Rubric:

- **2 points:** defining $v_q(x)$ correctly for $x \neq 0$,
- **1 point:** defining $v_q(x)$ correctly when $x = 0$,
- **2 points:** correctly dealing with the case when x or y is zero,
- **2 points:** recognizing that if $m = \min\big(v_q(x), v_q(y)\big)$ then $q^m \mid x$ and $q^m \mid y$,
- **2 points:** deducing that $q^m \mid x+y$ and thus $v_q(x+y) \geqslant m$.